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A second approximation for the velocity of a large gas bubble rising in an infinite liquid

By R. COLLINS

Department of Mechanical Engineering, University College London

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A second approximation for the velocity of a large gas bubble in an infinite liquid is derived from a linear perturbation of the first approximation. Previously known experimental results are in excellent agreement with the resulting velocity,

$$U = 0{\cdot}652(g\bar{a})^{\frac{1}{2}},$$

where $\bar{a}$ is the apparent radius of curvature of the front part of the bubble, and g the acceleration due to gravity. The shape of this second approximation is seen to be indistinguishable from spherical over a large region near the front stagnation point.

1. Introduction

In liquids which are effectively infinite in extent, a large gas bubble assumes the form which is sketched in figure 1. The upper surface of the bubble is virtually indistinguishable from spherical, the lower surface is unsteady, fluctuating about

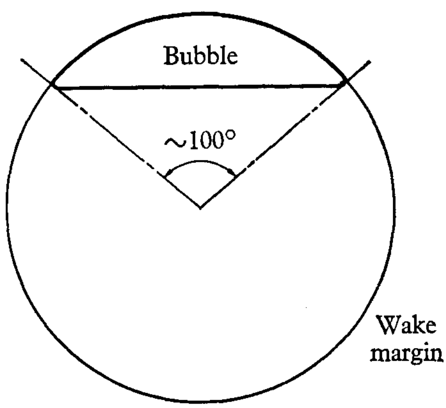

FIGURE 1. General features of a large gas bubble.

a horizontal plane, and the sphere which continues the bubble cap encloses a wake region behind the bubble. The detailed wake structure for the three-dimensional bubble has not been observed. Its extent is known, however, from the experiments of Davies & Taylor (1950) who were able to distinguish the wake region due to an optical anisotropy of nitrobenzene in which their bubbles were blown. For what is virtually the two-dimensional analogue of the bubble shown in figure 1, Collins (1965*a*) has shown the wake to consist of a large trailing vortex pair, as shown in figure 2 (plate 1). It is of some interest, and pertinent to a paper by Batchelor (1956) concerning wakes at large Reynolds number, that at values of the Reynolds number of the order of 10^4 the wake observed was closed, apparently stable and not cusped. In view of this it seems likely that, in three-dimensions, the bubble wake consists of a toroidal vortex, rather like Hill's spherical vortex.

The velocities of spherical-cap bubbles have been found by Rosenberg (1950) and by Davies & Taylor to be related to their apparent radii of curvature $\bar{a}$ through a relation of the form

$$U = k(g\bar{a})^{\frac{1}{2}}, \tag{1}$$

where g is the acceleration due to gravity and k a constant.

Rosenberg gave an experimental value of $k = 0{\cdot}645$, while the mean value of the results quoted in table 2 of the paper by Davies & Taylor gives $k = 0{\cdot}655$. In addition Davies & Taylor presented a theoretical evaluation, remarkable for its simplicity, which produced $k = \frac{2}{3}$. For many purposes the differences in these values would rightly be considered insignificant and the theoretical value of Davies & Taylor used. Their very closeness, however, coupled with the fact that the bubble shape cannot be truly spherical as assumed by Davies & Taylor, leads one to suspect that it might be worthwhile to seek a second approximation from a perturbation of the spherical form.

2. Formulation of the method

Consider the steady axisymmetric flow past the large bubble depicted in figure 3. Axes are fixed on the bubble which rises with velocity U and it is convenient to take the origin of co-ordinates at the centre of curvature of the bubble

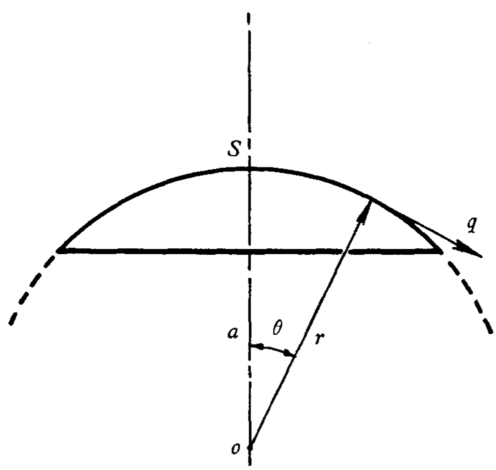

FIGURE 3. Model of the spherical cap bubble.

at its front stagnation point S. The upper surface of the bubble and its continuation enclosing the wake can be described by an equation of the form

$$r = a\{1 + f(\theta)\}, \tag{2}$$

where a is the radius of curvature at S, and the velocity distribution on this boundary may be written in magnitude as

$$q = Uh(\theta). \tag{3}$$

Large Reynolds and Weber numbers are known to characterize the motions of the bubbles considered here, so that, in determining the shape of the bubble cap, surface tension and viscous forces are usually considered to be negligible compared with gravity and inertia forces. The function $h(\theta)$ is consequently assumed to have the form appropriate to the irrotational flow past the body defined by equation (2). Equations (2) and (3) are to be regarded as modelling the flow over the upper surface of the bubble in some detail, and, to a lesser extent, the flow outside the attached wake previously discussed. There is no attempt to construct a detailed model of the wake itself, indeed it does not appear to be necessary to do so. Further discussion of this point, and of the assumptions implied in the method, is deferred until § 4.

If p_0 is the stagnation pressure, ρ the liquid density and p the pressure at a point (r, θ) on the bubble, then an application of Bernoulli's equation to a streamline on the bubble surface leads to a pressure coefficient distribution given by

$$c_p = \frac{p - p_0}{\frac{1}{2}\rho U^2} = \frac{2ga}{U^2} m(\theta) - \{h(\theta)\}^2, \tag{4}$$

in which

$$m(\theta) = 1 - \{1 + f(\theta)\}\cos\theta. \tag{5}$$

In order to ensure that the gas pressure within the bubble is constant, $m(\theta)$ and $h(\theta)$ should be such that $c_p = 0$ for all θ less than the maximum value defining the bubble rim.

Equation (4) is to be expanded as a power series in θ about $\theta = 0$, but before doing this there are some properties required of the functions $h(\theta)$ and $f(\theta)$ which are worth noting since they serve to eliminate many terms in the expansion. First, to preserve symmetry, $f(\theta)$ and hence $m(\theta)$, must be even functions; all their odd derivatives thus vanish at $\theta = 0$. Secondly, since a is the radius of curvature of the bubble at $\theta = 0$, it follows that $f_2(0) = 0$ and $m_2(0) = 1$.† Finally, bubble symmetry will produce $h(\theta)$ as an odd function so that all its even derivatives vanish at $\theta = 0$, and of course, $h(0) = 0$ because $(a, 0)$ is a stagnation point. Making use of these properties, the Taylor expansion of equation (4) gives

$$c_p = \left[\frac{ga}{U^2} - h_1^2\right]_{\theta=0} \theta^2 + \left[\frac{ga}{U^2}\frac{m_4}{4} - h_1 h_3\right]_{\theta=0} \frac{\theta^4}{3} + \left[\frac{ga}{U^2}\frac{m_6}{30} - \frac{h_1 h_5}{5} - \frac{h_3^2}{3}\right]_{\theta=0} \frac{\theta^6}{12} + \ldots, \tag{6}$$

and the condition of constant pressure will be achieved if all coefficients in the expansion are zero. Clearly only the exact (and unknown) bubble form will be capable of producing this result in equation (6), other assumed approximate forms will make only a finite number vanish. Nevertheless, the first coefficient in equation (6) always determines the velocity since, for any approximation, it gives

$$U = (ga)^{\frac{1}{2}} / h_1(0), \tag{7}$$

† To avoid a possible confusion of indices, a suffix will be employed in this section to denote differentiation with respect to the argument of a function. This coincides with the usual notation for series expansion of a function.

which may be used to eliminate ga/U^2 from all other coefficients. Labelling an approximation according to the number of coefficients that it eliminates in equation (6), the first is seen to be that of Davies & Taylor. On the basis of their experimental evidence showing the spherical-cap form they took $r = a = \bar{a}$ so that $f(\theta) = 0$. From the known irrotational flow past a sphere, $h(\theta) = \frac{3}{2}\sin\theta$, so that equation (7) gives

$$U = \tfrac{2}{3}(g\bar{a})^{\frac{1}{2}}, \tag{8}$$

which Davies & Taylor obtained using the same principles, but in a slightly different manner. For this approximation, the coefficient of θ^4 in equation (6) is $\frac{9}{16}$, and the form of the pressure coefficient c_p given by equation (4) is shown in figure 4.

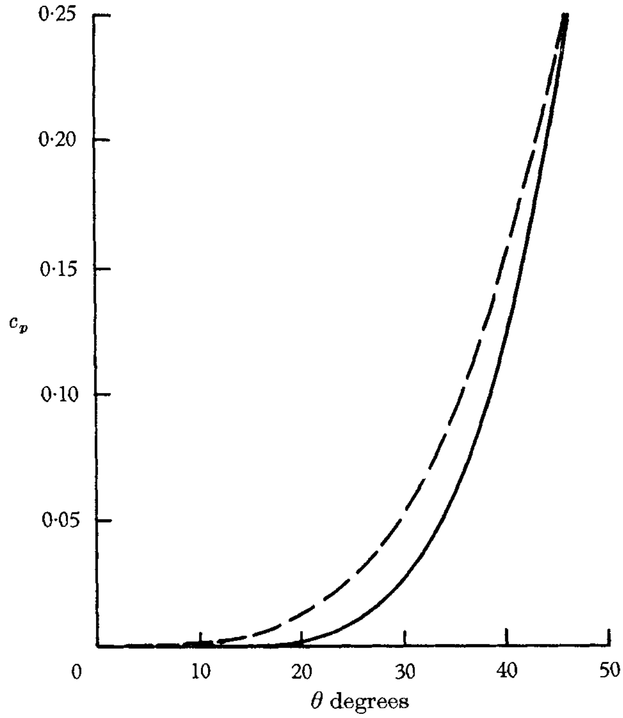

FIGURE 4. Variation of c_p with θ. – – – –, First approximation, Davies & Taylor; ———, second approximation, present theory.

Briefly, the method models the real flow in detail in a region which contains information about the terminal velocity, and then extracts this information from the model. It is semi-empirical to the extent that the first approximation relies on experimental evidence to suggest the otherwise unknown bubble form, and the second approximation is obtained from the first by a perturbation technique.

3. The perturbed bubble

Some of the properties required of $f(\theta)$ have already been listed in §2, they amount to the stipulation that $f(\theta)$ should be an even function with a leading term in θ^4 in its Taylor expansion. A simple function satisfying these conditions is

$$f(\theta) = -\epsilon \sin^4 \theta, \tag{9}$$

where ϵ is the perturbation parameter. By choosing a function with maximum value at $\theta = \frac{1}{2}\pi$, the perturbation parameter is efficiently utilized in modifying the boundary near the front stagnation point, and in addition, a bubble-wake system is produced with fore-and-aft symmetry, as suggested by figure 2 (plate 1). One is, of course, free to assume other forms for $f(\theta)$ in this method, for example a function giving a cusped tail to the wake to comply with Batchelor's proposal on the wake structure. This function would naturally be more complicated in form and would produce a correspondingly more complicated boundary condition in the perturbation problem. Further, it seems unlikely that this added refinement would significantly modify the flow over the frontal region, which is the region of prime interest here, and there is, as yet, no experimental evidence on which to base an estimate of the size of the cusped region relative to the rest of the wake. The merits of the function chosen here reside then in the fact that it is simple, plausible and adequately produces a good second approximation.

Since the flow is axisymmetric, it is possible to employ Stokes's stream function, ψ, which satisfies the equation (Lamb 1932)

$$D^2\psi = r^2 \frac{\partial^2 \psi}{\partial r^2} + \sin\theta \frac{\partial}{\partial \theta}\left\{\frac{1}{\sin\theta}\frac{\partial \psi}{\partial \theta}\right\} = 0, \tag{10}$$

solutions of which are

$$\left.\begin{aligned} \psi &= \frac{1}{n+1} r^{n+1} (1-\mu^2) \frac{dP_n}{d\mu}, \\ \text{and} \qquad \psi &= -\frac{1}{n} r^{-n} (1-\mu^2) \frac{dP_n}{d\mu}, \end{aligned}\right\} \tag{11}$$

where $\mu = \cos\theta$ and the P_n are Legendre polynomials. The perturbed shape will be defined by $\psi = 0$. For the basic flow, the stream function for the flow of a uniform stream past a sphere of radius a is†

$$\psi_0(r,\theta) = \tfrac{1}{2} U r^2 \sin^2\theta (1 - a^3/r^3), \tag{12}$$

and it will be assumed that the perturbed flow has a stream function of the form

$$\psi(r,\theta;\epsilon) = \psi_0(r,\theta) + \epsilon\psi_1(r,\theta) + \epsilon^2\psi_2(r,\theta) + \ldots. \tag{13}$$

In view of the close experimental agreement with the spherical first approximation, ϵ is considered small so that only linear terms will be retained in what follows. In the Appendix it is shown that the contribution of the second-order terms to $h_1(0)$ is negligible, thereby justifying their neglect.

On substituting into equation (10) and observing that the flow at infinity should be a uniform stream of velocity U, it is found that

$$\left.\begin{aligned} D^2\psi_0 &= 0, \quad \psi_0 \to \tfrac{1}{2} U r^2 \sin^2\theta \quad (\text{as } r \to \infty), \\ \text{and} \qquad D^2\psi_1 &= 0, \quad \psi_1 \to 0 \quad (\text{as } r \to \infty), \end{aligned}\right\} \tag{14}$$

† A numerical suffix is used here to distinguish the various stream functions.

while the boundary condition $\psi = 0$ on $r = a(1-\epsilon \sin^4\theta)$ gives

$$\psi_0\{a(1-\epsilon\sin^4\theta),\theta\}+\epsilon\psi_1\{a(1-\epsilon\sin^4\theta),\theta\}+\ldots = 0. \tag{15}$$

Assuming that ψ_1 is analytic in r, equation (15) may be expanded in a Taylor series about $r = a$ to give

$$\psi_0(a,\theta) - \epsilon a \sin^4\theta\psi_{0r}(a,\theta)+\epsilon\psi_1(a,\theta)+\ldots = 0. \tag{16}$$

Since $\psi_0(a,\theta) = 0$ from equation (12), equation (16) requires

$$\psi_1(a,\theta) = a\sin^4\theta\,\psi_{0r}(a,\theta), \tag{17}$$

which, using the basic solution in equation (12), gives

$$\psi_1(a,\theta) = \tfrac{3}{2}Ua^2\sin^6\theta. \tag{18}$$

The linear perturbation problem is therefore the solution of the equation

$$D^2\psi_1 = 0,$$

subject to the boundary conditions

$$\left.\begin{aligned} &\psi_1(a,\theta) = \tfrac{3}{2}Ua^2\sin^6\theta, \\ \text{and}\quad &\psi_1 \to 0 \quad \text{as} \quad r\to\infty. \end{aligned}\right\} \tag{19}$$

It is immediately obvious that only those solutions in equation (11) which contain inverse powers of r will satisfy the boundary condition at infinity. A convenient method of constructing the solution is to rewrite the boundary condition on the bubble as

$$\psi_1(a,\theta) = \tfrac{3}{2}Ua^2\sin^2\theta(1-2\cos^2\theta+\cos^4\theta), \tag{20}$$

to assume a series solution of the form

$$\psi_1 = -\Sigma\frac{A_n}{n}\frac{1-\mu^2}{r^n}\frac{dP_n}{d\mu}, \tag{21}$$

and to use the expansions of Legendre polynomials in powers of cos θ given by Hobson (1931) to pick out the coefficients A_n. The result is

$$\psi_1 = Ua^2\sin^2\theta\,[\tfrac{3}{2}\{a/r\}^5\{\cos^4\theta-\tfrac{2}{3}\cos^2\theta+\tfrac{1}{21}\}+\tfrac{2}{5}\{a/r\}^3\{1-5\cos^2\theta\}+\tfrac{36}{35}\{a/r\}], \tag{22}$$

and the stream function for the perturbed flow, correct to terms in ϵ, is then

$$\psi = \tfrac{1}{2}Ur^2\sin^2\theta(1-a^3/r^3)+\epsilon Ua^2\sin^2\theta \times [\tfrac{3}{2}\{a/r\}^5\{\cos^4\theta-\tfrac{2}{3}\cos^2\theta+\tfrac{1}{21}\}+\tfrac{2}{5}\{a/r\}^3\{1-5\cos^2\theta\}+\tfrac{36}{35}\{a/r\}]. \tag{23}$$

The component of the local radial velocity u along the bubble surface is $O(\epsilon^2)$ so that, for the linear problem, it does not contribute to the surface velocity distribution which is thus given by

$$q = v = \left[\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r}\right]_{r=a(1-\epsilon\sin^4\theta)} = U\sin\theta\{\tfrac{3}{2}-6\epsilon\cos^4\theta+8\epsilon\cos^2\theta-\tfrac{38}{35}\epsilon\}+\ldots, \tag{24}$$

where v is the velocity component perpendicular to the radius vector. Using equation (7) the bubble velocity becomes

$$U = 2(ga)^{\frac{1}{2}}/(3+\tfrac{64}{35}\epsilon). \tag{25}$$

It is now possible to determine the value of ϵ which will reduce the coefficient of

θ^4 in equation (6) to zero, thereby producing the second approximation. Making use of equation (7), the requirement is that

$$\{h_1 m_4 - 4h_3\}_{\theta=0} = 0, \tag{26}$$

which gives a quadratic for ϵ when the appropriate quantities are substituted. Discarding the squared term to be consistent, the necessary value is found to be $\epsilon = 7{\cdot}85 \times 10^{-2}$. The resulting bubble velocity from equation 25 is

$$U = 0{\cdot}636(ga)^{\frac{1}{2}}, \tag{27}$$

and for this approximation the variation of c_p with θ shown in figure 4 is naturally somewhat flatter over the front part of the bubble than that for the first.

4. Discussion

At first sight, the constant in equation (27) is lower than the experimental values of both Rosenberg and Davies & Taylor. It must be remembered, however, that the experimental results are expressed in terms of the apparent curvature of the cap, whereas the theoretical velocity is in terms of the curvature at the stagnation point. What is measured in practice is, of course, not the curvature *at* the stagnation point, but rather a mean value over a certain range of θ. Davies & Taylor carefully fitted circular arcs to their photographed bubble shapes and it appears that the total angle subtended at the apparent centre of curvature by that part of the bubble which matched these arcs was approximately 75°. They did not say whether the fit was achieved through a visual comparison or by numerical analysis of co-ordinates measured on the bubble cap. Rosenberg did not state the angle over which his own bubbles were fitted, but it is apparent from his list of references that he was aware of the work of Davies & Taylor and possibly fitted over the same angle. Referring to figure 5, a measure of the apparent radius of curvature of an arc of a bubble, SP, will be defined here as the distance SC, where C is the intercept of the perpendicular bisector of the chord SP with the axis of symmetry. Other definitions of $\bar{a}$ are certainly possible; they would all, however, give $\bar{a} < a$. From this definition, it follows that

$$\bar{a} = (a^2 - 2ar\cos\theta + r^2)/2(a - r\cos\theta). \tag{28}$$

For the bubble $r = a(1-\epsilon\sin^4\theta)$, if P is chosen at $\theta = 36°$, then the angle SCP, denoted by θ', is $37{\cdot}5°$, $\bar{a} = 0{\cdot}953a$, and the velocity becomes

$$U = 0{\cdot}652(g\bar{a})^{\frac{1}{2}}, \tag{29}$$

which is in excellent agreement with the experimental results of both Rosenberg and Davies & Taylor, as shown in figure 6. Since Rosenberg did not present tabulated data, this figure was drawn by superimposing Davies & Taylor's data and the theoretical lines on a magnified image of Rosenberg's own figure, which had been projected on a logarithmic grid. It may be further noted from this figure 6 that the two sets of experimental results contain roughly equal numbers of bubbles, that they cover different ranges of $\bar{a}$, and that, where they overlap, the two sets do not conflict.

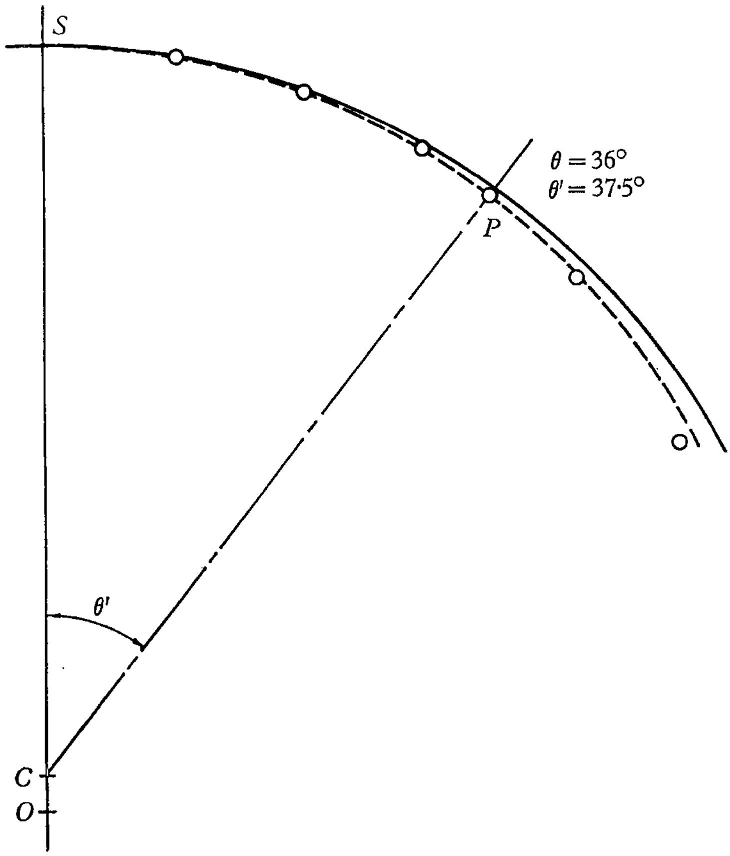

FIGURE 5. Comparison of bubble shapes: ———, $r = a$; ○, $r = a(1-\epsilon\sin^4\theta)$, $\epsilon = 0{\cdot}0785$; - - - , $\bar{a} = 0{\cdot}953a$.

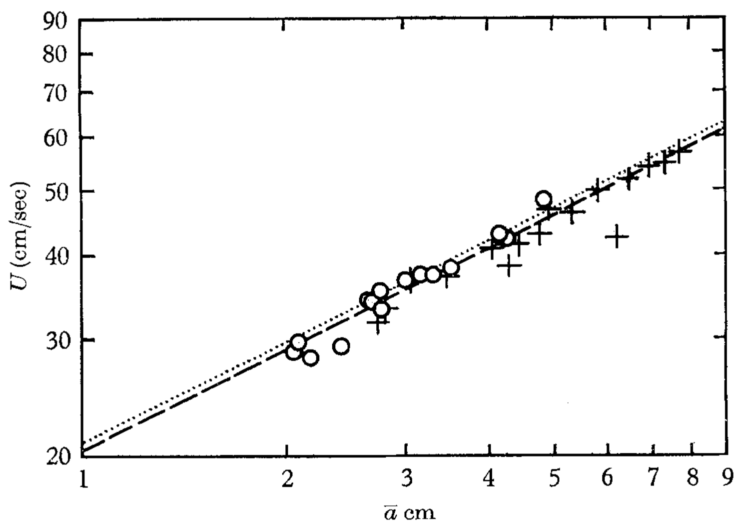

FIGURE 6. Comparison between experiment and theory. Experimental: +, Rosenberg; ○, Davies & Taylor. Theoretical: ..., first approximation, Davies & Taylor; – – –, second approximation, present theory.

Figure 5 compares the three curves $r = a$, $r = a(1-\epsilon\sin^4\theta)$ and $\bar{a} = 0{\cdot}953a$. Even when drawn on a large scale it is difficult to distinguish any difference between the latter two for values of $\theta' < 37{\cdot}5^\circ$ so that, over this range, the second approximation is visually indistinguishable from spherical. It should be noted that the first approximation has been plotted in figure 4 so as to compare c_p at the same point on the bubbles $r = a(1-\epsilon\sin^4\theta)$ and $\bar{a} = 0{\cdot}953a$. This has necessitated a slight adjustment in the abscissa for the first approximation which is strictly expressed in terms of the angle θ' in figure 5 rather than θ.

Some previous writers, for example Moore (1959) and Rippin (1959) have rejected closed wake models on the grounds that they have no drag by d'Alembert's paradox and therefore, presumably, an infinite terminal velocity. To state that closed wakes are inadmissible on these grounds, however, immediately begs a question: why does the present closed wake model, like that of Davies and Taylor,† give a finite velocity so close to experiment? First, a more comfortable feeling about this point is engendered by the realization that the present method does not calculate a drag coefficient for the bubble. This is because it does not relate the terminal velocity to the bubble volume, and hence to the buoyancy force producing the motion. The angle subtended by the bubble rim at the apparent centre of curvature is not theoretically determined, although it is possible that higher approximations obtained in this manner might shed some light on why the bubble terminates at $\theta' = 50^\circ$. But to answer the question it is necessary to re-examine the assumptions in the method. It should be appreciated that it is not necessary to assume that the fluid is inviscid; what is assumed is that the flow over the bubble cap is irrotational because viscous and surface tension forces are negligible in that region. It is not implied that the irrotational flow past the shape which continues the bubble cap and encloses the wake is a detailed model of the flow in that region, it merely models the gross feature of wake closure. This is what was meant by the statement in § 2 that the flow was modelled there only to some extent. Now d'Alembert's paradox deals with an inviscid fluid and may be derived either by considering overall changes in the momentum of a fluid stream flowing past a closed body, or by integrating the irrotational pressure coefficient distribution over the whole of the surface of the body. It thus has no relevance to the present method, although it would be relevant in aiming to solve the inviscid free boundary problem as was Rippin's numerical procedure. There is a paradox involved none the less: it is that, while the terminal velocity of the real bubble is determined by the rate of energy dissipation in the flow and primarily in the wake‡, it is not necessary to construct a detailed picture of the wake in order to deduce the relation between bubble shape and velocity. The explanation is that bubble velocity, bubble shape and the rate of energy dissipation are inter-related and in the sequence of events which determines the terminal velocity they appear, so to speak, on a closed loop.

† There is no explicit reference in the first approximation to the modelling of the closed wake, the modelling is implicit because the boundary assumed there was spherical.

‡ Davies & Taylor were able to estimate the rate of energy dissipation in the wake region and found it to be of the same order of magnitude as the rate of working of the buoyancy force.

Bubble shape and the energy dissipation are related because the structure of the whole flow is affected by the shape the bubble finally adopts. The rate of working of the buoyancy force may be equated quite generally to the rate of energy dissipation in the flow, thereby relating terminal velocity and energy dissipation. Finally to close the loop, bubble shape and velocity are related through the requirement of constant gas pressure within the bubble. The whole loop appears as

$$\begin{array}{c} \text{Shape} \\ \downarrow \\ \text{Dissipation} \\ \downarrow \\ \text{Velocity} \end{array}$$

and it is by a process of continual adjustment round the loop that the real bubble reaches its final shape and velocity. It can now be seen that by using the equation expressing the constant pressure requirement, bubble velocity can be inferred from an assumed shape which models the real shape and that this does not involve any consideration of the detailed wake structure. The region of prime interest in this procedure is the bubble cap, but it does seem reasonable to model the effect of wake closure, however crudely, on the flow in that region. There would be, in fact, little merit in trying to infer velocity from an assumed wake structure unless one could construct a model of the energy dissipation at least as accurate as the model of the cap.

5. Concluding remarks

The main result of the present analysis is to modify the value of $k = \frac{2}{3}$ found by Davies & Taylor to $k = 0{\cdot}652$, which is in slightly better accord with the experimental values. In addition, it is seen that a shape which satisfies the constant pressure requirement as far as terms in θ^4, is indistinguishable from a sphere over its frontal part. Perhaps the most useful consequence is that the first approximation of Davies & Taylor is thereby reinforced, its result being very close to the second. It would therefore be expected to give good results in more complicated geometries where the second approximation would become laborious. The two-dimensional problem when the liquid is finite in extent has been studied to the first approximation by Collins (1965*b*). An extension to the corresponding three-dimensional problem will be considered in a subsequent paper.

Finally, it is easily demonstrated that the present results apply equally well to Davidson's (1961) model of the bubble in a fluidized bed. An essential feature of his model is that the components of gas and particle velocities in the bubble surface are equal. This implies that the drag force on a particle moving on the bubble boundary is always normal to the direction of particle motion so that it moves as if it were on a smooth surface. Simple considerations of energy conservation show that

$$q^2 = 2qs, \tag{30}$$

where s is the distance of the particle below the front stagnation point, or in the terminology of § 2

$$2ga\, m(\theta)/U^2 - \{h(\theta)\}^2 = 0, \tag{31}$$

which is simply equation (4). The analysis subsequent to equation (4) thus applies to Davidson's bubble model.

Appendix

The boundary condition on the bubble for the second-order perturbation problem is obtained if the second-order terms are retained in the expansion of equation (15). The resulting perturbation problem is

$$D^2\psi_2 = 0,$$

subject to the boundary conditions

$$\psi_2 \to 0 \quad \text{as} \quad r \to \infty, \tag{32}$$

and
$$\psi_2(a,\theta) = a\sin^4\theta\psi_{1r}(a,\theta) - \tfrac{1}{2}a^2\sin^8\theta\psi_{0rr}(a,\theta).$$

Use of the basic and first-order solutions and a rearrangement of terms reduces the second boundary condition to

$$\psi_2(a,\theta) = -Ua^2\sin^2\theta\{\tfrac{15}{2}\cos^8\theta - 26\cos^6\theta + \tfrac{1123}{35}\cos^4\theta - \tfrac{566}{35}\cos^2\theta + \tfrac{181}{70}\}. \tag{33}$$

The solution of this problem is

$$\begin{aligned}\psi_2 = &-Ua^2\sin^2\theta\,[\tfrac{15}{2}\{a/r\}^9\{\cos^8\theta - \tfrac{28}{17}\cos^6\theta + \tfrac{14}{17}\cos^4\theta - \tfrac{28}{221}\cos^2\theta + \tfrac{7}{2431}\} \\ &-\tfrac{232}{17}\{a/r\}^7\{\cos^6\theta - \tfrac{15}{13}\cos^4\theta + \tfrac{45}{143}\cos^2\theta - \tfrac{5}{429}\} + 10{\cdot}1626\{a/r\}^5 \\ &\times\{\cos^4\theta - \tfrac{2}{3}\cos^2\theta + \tfrac{1}{21}\} - 4{\cdot}1516\{a/r\}^3\{\cos^2\theta - \tfrac{1}{5}\} + 1{\cdot}0908\{a/r\}],\end{aligned} \tag{34}$$

which was obtained in the same manner as the first-order solution and which is approximate in the sense that some of the more unwieldy fractions have been replaced by decimals rounded to four places (for example, 1·0908 is the ratio of two terms $O(10^6)$).

With both ψ_1 and ψ_2 determined, it is now possible to calculate the surface speed, and hence $h_1(0)$ correct to terms in ϵ^2 from equation (13). It might be thought necessary to include in this calculation the contribution from the component of u which is $O(\epsilon^2)$ but in fact, since u varies directly with $\sin^4\theta$, it does not contribute to $h_1(0)$. As in the linear case $h_1(0)$ may therefore be calculated from v, resulting in the equation

$$h_1(0) = 1{\cdot}5(1 + 0{\cdot}6095\epsilon - 0{\cdot}1420\epsilon^2\ldots). \tag{35}$$

It is apparent that, with $\epsilon = 0{\cdot}0785$ from the linear perturbation, the term in ϵ^2 modifies $h_1(0)$ by only 0·08 %. The neglect of the second-order term in § 3 is thus considered to be justified.

Note added at the proof stage

Dr Davidson has kindly pointed out some related work on this topic by Temperley & Chambers (1945). Their experiments led them to conclude that the velocities of spherical-cap bubbles having cap radii up to 15 cm were consistent with a value of $k = \frac{2}{3}$. It must be noted, however, that their measurements of $\bar{a}$ were accurate only to 20 %, so that they would not have been able to detect the slight change brought about by the present second approximation. They also attempted to improve Davies & Taylor's result by assuming that the flow could be modelled with a combined source and doublet in a uniform stream, with which it is possible to produce a second approximation. Expressed in the present notation their results give $U = 0{\cdot}535(ga)^{\frac{1}{2}} = 0{\cdot}54(g\bar{a})^{\frac{1}{2}}$, where $\bar{a}$ has been fitted as in §4. (The first result quoted appears rounded-off in their paper to $U = 0{\cdot}54(ga)^{\frac{1}{2}}$.) As they noted, this worsens the agreement with experiment, a result almost certainly brought about by the presence of the source term producing an infinite wake. Recent observations of spherical-cap bubble wakes by the writer support the conjecture on wake structure advanced in §1 of this paper.

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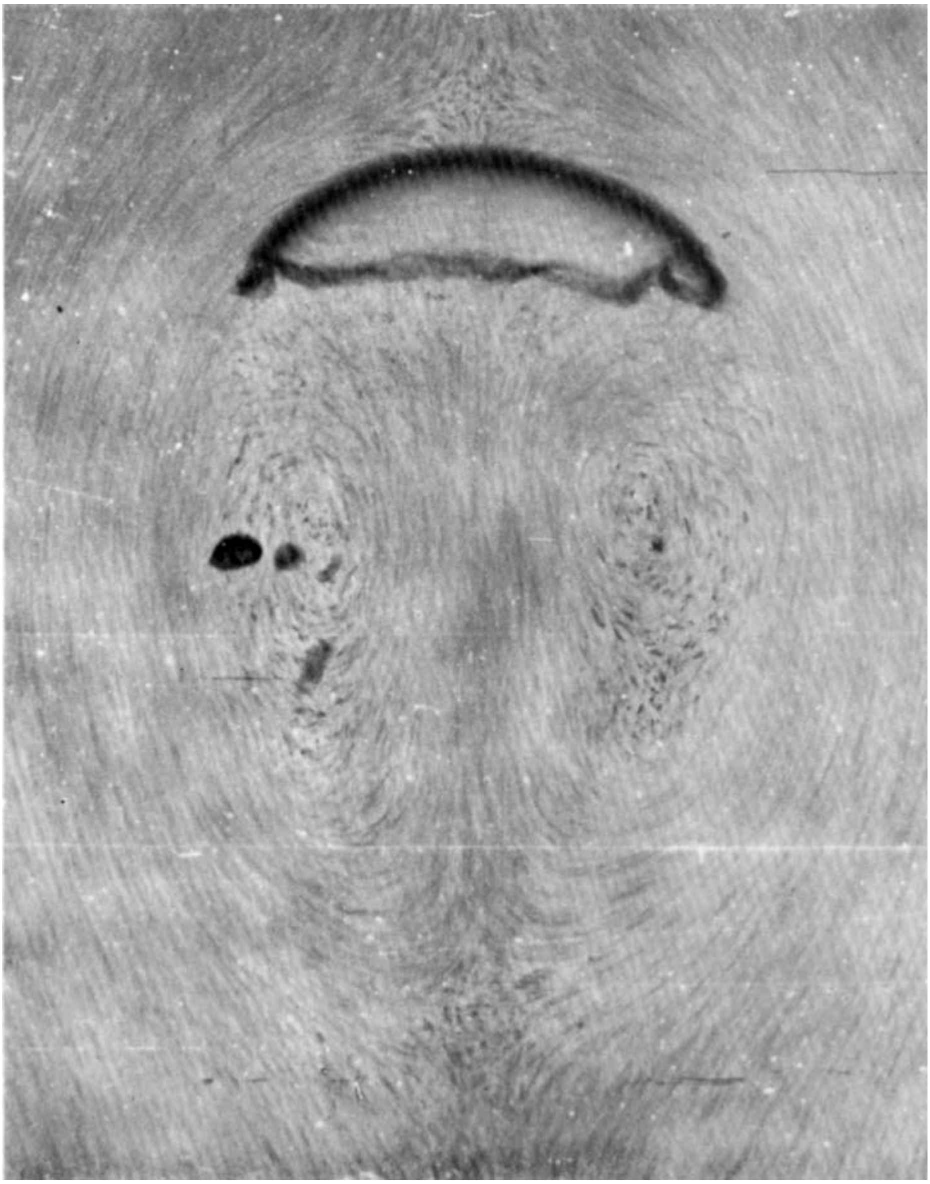

FIGURE 2. Wake structure for a large two-dimensional gas bubble.